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THE QUASIGEOID AND THE STANDARD ELEVATIONS

- EAST GERMANY -

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F O R E W O R D

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(Conclusion from No. 2, 1960)

6. Bruns' Theorem

Taking point A on geoid $W = W_0$ then, according to equation
(29a), the following equation holds for point B of the topographic
earth surface

$$W_B - W_0 = -\int_A^B g\,dh. \tag{31}$$

For the same point B, the normal potential U_B then looks as fol-
lows:

$$U_B - U_0 = -\int_0^B \gamma\,dh = -\int_0^{B'} \gamma\,dh - \int_{B'}^B \gamma\,dH. \tag{32}$$

taking model earth as reference ellipsoid then is (see Fig. 3)

According to (16) we have $\quad -\int_0^{B'} \gamma\,dh = U_{B'} - U_0$

and according to (15), we have $\quad U_{B'} - U_0 = W_B - W_0$

From (16) and (15) follows $W_B - W_0 = -\int_0^{B'} \gamma\,dh.$

Using (31), we get $\quad -\int_0^{B'} \gamma\,dh = -\int_A^B g\,dh$

and from (32) follows

$$U_B - U_0 = -\int_A^B g\,dh - \int_{B'}^B \gamma\,d\bar{H}. \tag{33}$$

For the last integral, however, one can take γ as constant and,
because of the meaning of γ , we can set up the following:

$$\int_{B'}^B \gamma\,d\bar{H} = \gamma \int_{B'}^B d\bar{H} = \gamma\zeta$$

Thus we get $\quad U_B - U_0 = -\int_A^B g\,dh - \gamma\zeta,$ $\tag{34}$

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From (31) and (34) we get the following by forming the difference:

$$W_B - U_B - (W_o - U_o = \gamma \zeta \tag{35}$$

or

$$\zeta = \frac{W_B - U_B}{\gamma} - \frac{W_o - U_o}{\gamma} . \tag{36}$$

If we define the perturbation function T_B as follows:

$$T_B = W_B - U_B \tag{37}$$

and if we consider that the model earth, which defines the quasi-geoid, has been chosen because of its property 5 in such a way that $W_o = U_o$, then we finally get the well-known theorem by Bruns, after dropping the index B, using

$$\zeta = \frac{T}{\gamma} \tag{38}$$

But here γ is the normal gravity on the physical earth surface and ζ the altitude of the quasigeoid.

7. Development of an Integral Equation for the Altitude of the Quasigeoid

According to a calculation by Levallois /2/, we now want to develop an integral equation for ζ ; this equation was obtained by Molodjenski /6/ in a different way and was mentioned by Sagrebin /10/.

Let us start from Green's theorem

$$\iint_S \left(Y \frac{\partial V}{\partial n} - V \frac{\partial Y}{\partial n} \right) dS = \iiint_\gamma \left[Y \cdot \Delta V - V \cdot \Delta Y \right] d\gamma , \tag{39}$$

where γ is a volume with the surface S, and Y and V are two harmonic functions of space and $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is the Laplace operator.

Here we take for Y the gravitation potential W of the earth and put $V = 1:r$, where r is the distance of the moving point P on the surface S from a fixed point M on this surface. Let the volume extend into infinity. The direction of the outer normal n of the volume τ points from the point P to the earth. If we separate point M, which lies on S, from the points of the volume τ , by means of a small hemisphere Σ , placed on S, around M as center with infinitesimal radius ς , then the inner boundary of volume consists of the surface S, let this surface be S' -- minus the small circle that has been drawn with the radius S around M -- and of the surface of the hemisphere Σ around M.

In this region τ , W and 1:r are harmonic functions; for them, the Laplace expression thus becomes $\Delta W = 0$ and $\Delta \frac{1}{r} = 0$.

The right side of Green's theorem (39) disappears too and one obtains

$$\iint \left[w \frac{\partial \frac{1}{r}}{\partial n} - \frac{1}{r} \frac{\partial w}{\partial n} \right] dS' +$$

$$+ \iint_{\Sigma} \left[w \frac{\partial \frac{1}{\varrho}}{\partial n} - \frac{1}{\varrho} \frac{\partial w}{\partial n} \right] d\Sigma = 0.$$

$$(40)$$

For the surface element $d\Sigma$, one obtains in polar coordinates:

$$d\Sigma = \varrho^2 \sin \vartheta \; d\vartheta \; d\lambda \tag{41}$$

and for the hemisphere we furthermore get:

$$\frac{\partial \frac{1}{\varrho}}{\partial n} = \frac{- \partial \frac{1}{\varrho}}{\partial \varrho} = + \frac{1}{\varrho^2} .$$

$$(42)$$

The following is valid for ϑ and λ of the hemisphere Σ

$$-\frac{\pi}{2} \leq \vartheta \leq +\frac{\pi}{2} \quad \text{and} \quad 0 \leq \lambda \leq \pi . \tag{43}$$

From this then follows

$$\iint_{\Sigma} \left(w \frac{\partial \frac{1}{\varrho}}{\partial n} - \frac{1}{\varrho} \frac{\partial w}{\partial n} \right) d\Sigma$$

$$= \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \int_{0}^{\pi} \left(+ w \frac{1}{\varrho^2} - \frac{1}{\varrho} \frac{\partial w}{\partial n} \right) \varrho^2 \sin \vartheta \, d\vartheta \, d\lambda$$

$$= + \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \int_{0}^{\pi} w \sin \vartheta \, d\vartheta \, d\lambda - \int_{-\frac{\pi}{2}}^{+\frac{\pi}{2}} \int_{0}^{\pi} \varrho \frac{\partial w}{\partial n} \sin \vartheta \, d\vartheta \, d\lambda$$

$$(44)$$

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For lim $\zeta = 0$ we get from (40) and (44)

$$\lim_{Q=0} \int\int_{S'} \left[W \frac{\partial \frac{1}{r}}{\partial n} - \frac{1}{r} \frac{\partial W}{\partial n} \right] dS'$$

$$+ \lim_{\zeta=0} \int\int_{\Sigma} \left[W \frac{\partial \frac{1}{\zeta}}{\partial n} - \frac{1}{\zeta} \frac{\partial W}{\partial n} \right] d\Sigma$$

$$= \int\int \left(W \frac{\partial \frac{1}{r}}{\partial n} - \frac{1}{r} \frac{\partial W}{\partial n} \right) dS + \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^n \lim_{Q=0} W \sin \vartheta \, d\vartheta \, d\lambda$$

$$- \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{n} \lim_{Q=0} \zeta \frac{\partial W}{\partial n} \sin \vartheta \, d\vartheta \, d\lambda = 0.$$

(45)

Herein now $\lim_{S=0} W = W_M$ and the second integral becomes

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^n \lim_{Q=0} W \sin \vartheta \, d\vartheta \, d\lambda = W_M \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin \vartheta \, d\vartheta \int_0^n d\lambda = -2\pi W_M.$$

The last integral in (45) vanishes with $\lim \zeta = 0$. Using this, we obtain from (45)

$$\int\int_S \left\{ W \frac{\partial \frac{1}{r}}{\partial n} - \frac{1}{r} \frac{\partial W}{\partial n} \right\} dS - 2\pi W_M = 0.$$

(46)

Finally one obtains from this

$$W_M = \frac{1}{2\pi} \int\int_S \left\{ W \frac{\frac{1}{r}}{\partial n} - \frac{1}{r} \frac{\partial W}{\partial n} \right\} dS.$$

(47)

Applying the same formula to the normal potential U -- this is permissible, since U satisfies all conditions of Green's theorem -- one then obtains in an analogous way

$$U_M = \frac{1}{2\pi} \int\int_S \left\{ U \frac{\partial \frac{1}{r}}{\partial n} - \frac{1}{r} \frac{\partial U}{\partial n} \right\} dS.$$

(48)

Comparing the formulas (47) and (48), we obtain

$$W_M - U_M = T_M$$

$$= \frac{1}{2\pi} \iint_S \left[(W-U) \frac{\partial \frac{1}{r}}{\partial n} - \frac{1}{r} \left(\frac{\partial W}{\partial n} - \frac{\partial U}{\partial n} \right) \right] dS. \tag{49}$$

According to Bruns' theorem (38) we now have $T = \zeta \gamma = W-U$ and one obtains

$$\zeta M \gamma = \frac{1}{2\pi} \iint_S \left[\zeta \gamma \frac{\partial \frac{1}{r}}{\partial n} - \frac{1}{r} \left(\frac{\partial W}{\partial n} - \frac{\partial U}{\partial n} \right) \right] dS. \tag{50}$$

In the integrand $\partial W/\partial n$ and $\partial U/\partial n$ are the derivatives of W and U in the direction of the normal on the topographic earth surface. In the moving point P of the physical earth surface S, the latter's normal forms, with the directions of the fields $\vec{g}$ or $\vec{\gamma}$, the angles α or $\alpha' - \alpha$, respectively. Then, because of (4) or (13) and considering the relations in Figure 6, we get

$$\frac{\partial W}{\partial n} = \frac{\partial W}{\partial H} \cdot \frac{\partial H}{\partial n} = - g_P \cos x$$

$$\frac{\partial U}{\partial n} = \frac{\partial U}{\partial H} \cdot \frac{\partial H}{\partial n} = - \gamma_P \cos \alpha' = - \gamma_P \cos x. \tag{51}$$

Using (51), equation (50) then goes over into

$$\zeta M \gamma = \frac{1}{2\pi} \iint_S \left[\zeta \gamma \frac{\partial \frac{1}{r}}{\partial n} + \frac{1}{r} (g_P - \gamma_P) \cos x \right] dS. \tag{52}$$

According to free-air reduction (14), the following equation holds for the altitude $H = \zeta$ of P over the model earth as reference surface:

$$\gamma_P = \gamma_k \left(1 - 2 \frac{\bar{H} + \zeta}{R} \right), \tag{53}$$

One therefore obtains

$$\zeta M \gamma = \frac{1}{2\pi} \iint_S \left[\zeta \gamma \frac{\partial \frac{1}{r}}{\partial n} \right. $$

$$\left. + \frac{1}{r} \left\{ g_P - \gamma_k \left(1 - 2 \frac{\bar{H} + \zeta}{R} \right) \right\} \cos x \right] dS. \tag{54}$$

If we want to introduce into this equation the "free air anomaly" Δ g, we must reduce g_P with the altitude of P to the quasigeoid, or -- which amounts to the same thing, except for second order magnitudes -- relate χ_k to the level of g_P. Using (14) we get

$$\Delta g = g_P -- V_P = g_P -- V_k \left(1 -- 2 \frac{\bar{H}}{R} \right) . \tag{55}$$

Finally we obtain the desired integral equation

$$\delta V = \frac{1}{2\pi} \iint\limits_S V \frac{\partial \frac{1}{r}}{\partial n} \, dS \neq \frac{1}{2\pi} \iint\limits_S \frac{\Delta g}{r} \cos x \, dS$$

$$\neq \frac{1}{2\pi} \iint\limits_S \frac{2V \delta}{Rr} \cos x \, dS. \tag{56}$$

This equation gives the quasigeoid as function of the free air anomaly on the topographic earth surface and determines the value δ as distance of the quasigeoid from the reference ellipsoid.

The solution of the integral equation (56) should be possible by iteration if one knows enough values of the gravitation anomaly on S. A solution in first approximation for the integral equation (56) one obtains, as Molodjenski showed in his work, from the Stockes formula

$$\delta_M = \frac{R}{2\pi V_0} \int_0^{\pi} \int_0^{2\pi} (g --V) F(\psi) \, d\psi \cdot d\alpha$$

Here we have

R — a mean earth radius;

χ_0 — a mean normal gravity on the earth;

g — the measured value of the gravity in the moving point P;

χ — the normal value of the gravity which corresponds to the measured value g; it is calculated with the normal altitude H and the latitude φ of the point P;

ψ — the angular distance of the point P from the fixed point M which is to be investigated; it runs from 0 to 2π;

α — the azimuth of the arc MP; it runs from 0 to π;

$F(\psi)$ — the well-known Stockes function

$$F(\psi) = \frac{1}{2} \sin \psi \quad \operatorname{cosec} \frac{\psi}{2} -- 6 \sin \frac{\psi}{2} \neq 1 -- 5 \cos \psi$$

$$-- 3 \cos \psi \quad \ln \left\{ \sin \frac{\psi}{2} \neq \sin^2 \frac{\psi}{2} \right\} .$$

A derivation of the Stockes formula without the use of spherical harmonics (functions) can be found in Baeschlin [12]. For practical applications, this formula was prepared by W. F. Jeremejew [13].

The numerical evaluation of the integral is made, using the same process as P. S. Sakatow uses it in his book [14] for the calculation of the components of the plumb deviations. Better values are obtained by using the palettes of Jeremejew and Jurkina [15] with which one can calculate ζ up to a few dm. This is a quite satisfactory result for the geoidal part of the altitude.

We must stress again that both altitude portions can be calculated exactly. This cannot be done for the orthometric altitudes.

8. Conclusion

It is quite justified to ask for an exact solution for the calculation of the distance of a point on the physical earth surface from the reference ellipsoid. One must consider that all measurements of geodetic quantities are performed on the physical earth surface. This surface, however, is not suitable for relating on it the measured values as far as their interdependence is concerned. For this purpose, the measured elements of geodetic grids, e.g., base lines, astronomic coordinates, horizontal directions, etc., have until now been reduced to sea level, i.e., to the geoid. From the geoid the quantities were transferred to the reference ellipsoid. Distances in this procedure were transferred, under conservation of the magnitude, from the geoid onto the reference ellipsoid, neglecting the difference in curvature of both surfaces. This procedure is not exact mathematically and adds further uncertainties to the uncertainties connected with the geoid, so that the value of precise measurements can be strongly reduced. This procedure therefore must be rejected. A method that avoids these errors is the "projection procedure" developed in the USSR. In this procedure, the measured geodetic quantities are projected along the lines of force to the quasigeoid; they are then projected along plumb directions to the reference ellipsoid. As Pizetti suggested, the projection is thus performed in two steps. A confrontation of both methods is given by Milan Bursa [17]. The Pizetti projection, with the quasigeoid as intermediate surface, can be performed without the use of any hypothesis. Here, all the quantities that are normally neglected are perfectly known. This clearly shows the great importance of the idea of M. S. Molodjenski.

If one takes into consideration finally that the quasigeoid coincides on the free oceans with the geoid of Listing and that it touches it on the shores of the continents -- i.e., here only the radii of curvature differ -- and if one further considers that the

quasigeoid has only a small distance from the geoid, then one can understand the suggestion made by Levallois /9/ that the quasigeoid be defined as earth figure. One can do this especially since its surface can be determined with satisfactory exactness from the integral equation that was developed above, if one only knows sufficient values of the anomaly spaced equally over the whole earth. At least this definition would be no less conventional than the geoid used by Listing, and the equations used for the quasigeoid are exact. If in the future the dimensions of the model earth (Krassowski ellipsoid) and, with this, the dimensions of the quasigeoid should change slightly, then all obtained values could be transformed exactly into the new values. This is a further advantage of the quasigeoid.

All these facts caused the Soviet geodesists to introduce for the USSR the normal altitudes and thus the quasigeoid.

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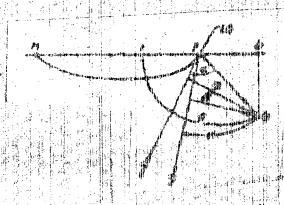
FIGURE APPENDIX

Figure 6

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